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Railroad discounting ☆

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HIGHLIGHTS

- One-point spatial perspective implies a decreasing "spatial discount rate".
- Our spatial view of the world corresponds to hyperbolic, not constant discounting.
- The continuous limit corresponds to logarithmic discounting.

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ABSTRACT

Spatial perspective implies a hyperbolic spatial discount rate. To the extent that discounting with respect to space and to time are analogous, this result provides further evidence that hyperbolic discounting is a good description of how we view the world.

1. Introduction and result

Economists use constant discounting in most dynamic models, because this is the only form of discounting that leads to time consistency of optimal plans. Constant discounting is certainly familiar, i.e., customary in our profession, but it is not clear whether it is part of the more general cultural DNA. Ramsey (1928) remarked "My picture of the world is drawn in perspective. . . . I apply my perspective not merely to space but also to time." Perspective applied either to space or to time does seem part of our cultural DNA.

Perspective means that distant objects appear smaller, but beyond that it is not clear whether it is consistent with constant or hyperbolic discounting, or something entirely different. This note shows that perspective applied to space corresponds to hyperbolic discounting. This result is significant as a reminder that the widespread adoption of constant discounting is due to its time consistency, and not necessarily because it provides a good description of how people see the world. Many other papers have discussed the plausibility of hyperbolic discounting, both in the context of individual decision problems and for societal problems such as climate change (Laibson, 1997; Barro, 1999; Cropper and Laibson, 1999; Heal, 1998, 2001; Harris and Laibson, 2001; Karp, 2005; Karp and Tsur, 2011). I do not review those arguments, because my objective here is merely to establish that, to the extent we accept the analogy between space and time, our view of the world corresponds to hyperbolic, not exponential, discounting.

I establish this claim using a model of one-point perspective. Fig. 1 shows a long (possibly infinite) railroad in one-point perspective. The rails, which are parallel in reality, appear to converge at the horizon. The horizontal lines, the railroad ties (hereafter "ties"), are actually evenly spaced, but more distant ties appear to be closer together and shorter. The letters $A, B, C, \ldots$ denote both the successive ties, and also their apparent length. The diagonals between ties, the dashed lines, identify by their intersection the location of the tie midway between any two ties, in actual (as distinct from apparent) distance.

The person looking at this railroad is standing in front of the first tie, A. If this person were ubiquitous, she would correctly perceive

☆ I thank Niklas Mattson for suggesting an argument for a proof of a special case of Proposition 1. Although not used in this paper, that special case was instrumental in forming my views of the general case considered here. I also thank Terry Iverson for comments on an earlier draft. The usual disclaimer holds.

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the rails to be parallel and the ties to be evenly spaced and of equal length. Because she is located in a particular position, the rails appear to converge and the ties to get closer together and shorter in the distance. The taller she is, the more her view resembles that of the floating deity. The height of the triangle is Q and the apparent distance between the ties E and A, is αQ.

I require some notation in order to state the proposition. I use

$L_0, L_1, L_2 \dots L_{n-1}, L^{\text{anchor}}$

(instead of $A, B, C, \dots$) to denote the consecutive ties and their apparent length. I fix the tie L^{anchor} in physical space; for example, it is the tie located one mile from the first tie, L_0. In Fig. 1 we can identify tie E with L^{anchor}. The number of ties from the first tie to the anchor is n. Continuing the space–time analogy, the distance between L_0 and L^{anchor} corresponds to the amount of time from today until a future event; n is the number of intervals between the fixed total distance (or time). The parameter n is key in providing a general result. Increasing n is analogous to measuring time in days instead of years.

The parameter α is a measure of perspective. An extremely tall person would correctly perceive that the distance between L_0 and L^{anchor} comprises only a small part of this long railroad. For this person, $\alpha \approx 0$. For a person with eyes close to the ground, it appears that the distance between L_0 and L^{anchor} comprises nearly the entire railroad. For this person, $\alpha \approx 1$. Thus, a larger value of α corresponds to higher spatial discounting. The actual distance between a particular tie and L_0 is the spatial analog of the amount of time between the present and some future date. The fact that more distant ties appear shorter, to a person at L_0, is the spatial analog of the idea that events that are distant in time should receive less weight in a welfare calculation.

With this analogy, I define $\beta_i^{(n)} = \frac{L_i^{(n)}}{L_{i-1}^{(n)}}$ for $i = 1, 2, 3, \dots, n$ and $\beta_0 = \frac{L_0}{L_0} = 1$, as the spatial discount factor between two consecutive ties, from the standpoint of the person at L_0. The corresponding discount rate is $r_i^{(n)} = 1 - \frac{1}{\beta_i^{(n)}}$. I find the formula for $r_i^{(n)}$, and use that to show that $r_i^{(n)}$ decreases in i. In this sense, perspective with respect to space implies hyperbolic discounting. The superscript (n) is a reminder that the apparent length of the i'th tie, and thus the corresponding discount factor and rate, depend on the actual proximity of the ties: e.g. whether they are spaced every foot or every meter. I normalize by setting $L_0^{(n)} = 1$ for all n. This normalization and the definition of β_i imply

$$L_i^{(n)} = \Pi_{j=1}^{i} \beta_j^{(n)}. \tag{1}$$

Thus, $L_i^{(n)}$ is the spatial analog of the present value of the i'th tie.

Define $M = \{2, 4, 8, 16, \dots\}$, the set of integer powers of 2.

Proposition 1. *For $n \in M$, the spatial discount factor is*

$$\beta_k^{(n)} = \frac{n - (n + 1 - k)\,\alpha}{n - (n - k)\,\alpha} \tag{2}$$

and the spatial discount rate is

$$r_k^{(n)} = \frac{\alpha}{n - (n + 1 - k)\,\alpha} \tag{3}$$

for $1 \le k \le n$. With this formula

$$\begin{aligned} \frac{dr_k^{(n)}}{dk} &= -\frac{\alpha^2}{(n - \alpha + k\alpha - n\alpha)^2} < 0 \\ \frac{dr_k^{(n)}}{d\alpha} &= \frac{n}{(n - \alpha + k\alpha - n\alpha)^2} > 0. \end{aligned} \tag{4}$$

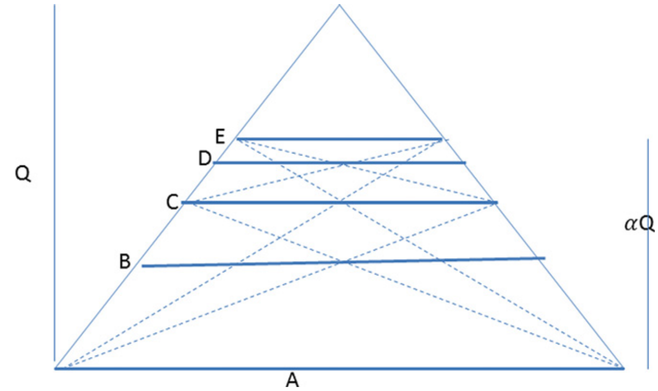

Fig. 1. A railroad in one-point perspective. The horizontal lines are the railroad ties and the sides of the triangle are the tracks, converging at the horizon. The height of the triangle is Q (not an axis) and the height of tie E is αQ. The letters A, B, C, D and E denote both the ties and their apparent length.

The discount rate decreases with respect to space (spatial hyperbolic discounting) and increases as the perspective gets "closer to the ground" (larger α).

The results are slightly easier to interpret if we consider the continuous space model. Define D as the actual distance between the first tie and the anchor, L^{anchor}, and use s to denote the actual distance between the first tie and an arbitrary tie. Taking the continuous space limit of the discrete space discount factor and discount rate gives the following result.

Proposition 2. *In the continuous space model with spatial index s, the discount factor and discount rate corresponding to position s, from the perspective of the agent at $s = 0$, are*

$$\begin{aligned} L(s) &= \frac{D(1-\alpha)}{D(1-\alpha) + s\alpha} \quad \textit{and} \\ r(s) &= -\frac{\frac{dL(s)}{ds}}{L(s)} = \frac{\alpha}{D(1-\alpha) + s\alpha}. \end{aligned} \tag{5}$$

The "Weber–Fechner law" states that human response to a change in stimulus, such as sound or light, is inversely proportional to the pre-existing stimulus. Heal (2001) notes that this law, if applied to discounting, implies that the discount factor (a function of time, t) is of the form t^{-K}, where K is a positive constant. He calls this "logarithmic discounting". The spatial analog is s^{-K}.

Defining $\kappa = \frac{\alpha}{D(1-\alpha)}$ and $\tilde{s} = 1 + \kappa s$, gives $L\left(\tilde{s}(s)\right) = \tilde{s}^{-1}$, a special case of logarithmic discounting (with $K = 1$) and an altered time scale. The discount rate,

$$r(s) = \frac{\kappa}{\tilde{s}^2 L\left(\tilde{s}(s)\right)} = \frac{\kappa}{1 + \kappa s}, \tag{6}$$

depends on distance s and a single parameter, κ. A smaller value of α or a larger value of D, both of which reduce κ, correspond to an agent who resembles more the ubiquitous deity, and less the person whose eyes are close to the ground. Smaller α or larger D correspond to a higher spatial discount factor. Although the discount rate (and factor) depend only on κ, knowing either the height of the triangle (Q) or the actual distance to the anchor (D) is not enough to determine that parameter. We need to know both (or some other combination of two parameters). By its construction, one-point perspective always looks like a triangle. A smaller Q means that we see distances less sharply, i.e. the discount rate is larger. But to be able to interpret the exact effect of a smaller Q on our perspective, we need a point of reference in actual space: D.

Iverson (2013) uses an extension of Golosov et al. (2014) to study climate policy under non-constant discounting; see

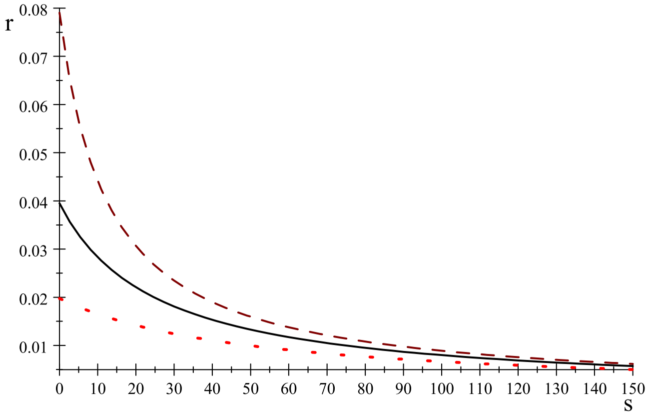

Fig. 2. Discount rates for $\kappa = 3.95 \times 10^{-2}$ (solid) and for this value doubled and halved (dash and dotted).

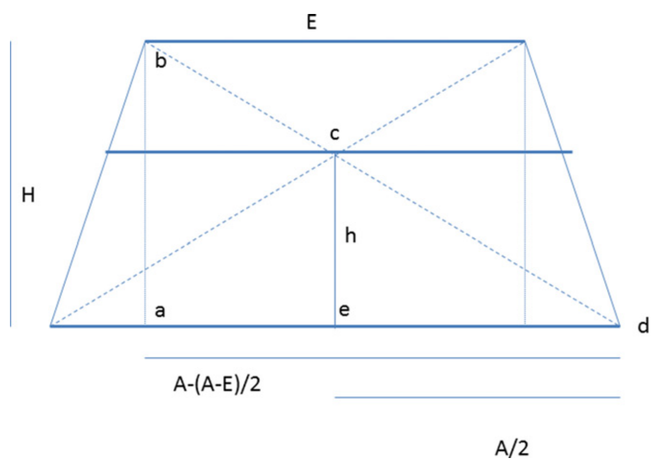

Fig. 3. The triangles *abd* and *ecd* are similar. Their respective heights are H and h and their respective bases are $A - \frac{A-E}{2}$ and $\frac{A}{2}$.

also Gerlagh and Liski (2012). This model is linear in state variables (after a transformation), making it possible to obtain the equilibrium savings rate in closed form, as a function of discounting parameters and the share of capital in production. Using the discount rate in Eq. (6), but expressed as a function of time (years) rather than distance, and a savings' rate of 22%, Iverson's formula implies $\kappa = 3.95 \times 10^{-2}$. Fig. 2 shows the graph of $r(s)$ for this value of κ and for twice and half this value.

2. Proof of the propositions

Proposition 1. The restriction that $n \in M$ makes it easy to use an inductive proof. Eq. (3) follows from Eq. (2), and Eqs. (4) are the result of taking derivatives, so I need only confirm Eq. (2). After stating some facts from elementary geometry, I begin the induction with $n = 2$, the first element of M.

For an isosceles triangle with height Q and base A, the length of any line E parallel to A at height H, with endpoints on the sides of the triangle, is

$$E = A\left(\frac{Q-H}{Q}\right) = A(1-\alpha). \tag{7}$$

The first equality follows from the property of similar triangles and the second uses $\alpha = \frac{H}{Q}$.

Let A be one tie and E an arbitrary subsequent tie, with apparent distance between them equal to H. Relative to A, the apparent height of the intersection of their diagonals (the dashed lines in Fig. 1) is h, the solution to

$$\frac{h}{\frac{A}{2}} = \frac{H}{A - \frac{(A-E)}{2}} \Longrightarrow h = A\frac{H}{A+E}. \tag{8}$$

The first equality follows from the property of similar triangles, and is evident from Fig. 3. The length of the tie through this point of intersection, with endpoints on the sides of the trapezoid formed by joining the endpoints of A and E, is C. The length of any segment parallel to A, connecting the sides of the trapezoid, is a convex combination of A and E. This fact implies

$$\begin{aligned} C &= \left(1 - \frac{h}{H}\right)A + \frac{h}{H}E = \left(1 - \frac{A\frac{H}{A+E}}{H}\right)A + \frac{A\frac{H}{A+E}}{H}E \\ &= 2A\frac{E}{A+E} = 2A\frac{A(1-\alpha)}{A + A(1-\alpha)} = 2A\frac{1-\alpha}{2-\alpha}. \end{aligned} \tag{9}$$

I begin the inductive chain using these results and the definition of $\beta_i^{(n)}$. They imply

$$\beta_1^{(2)} = \frac{C}{A} = \frac{2A\frac{1-\alpha}{2-\alpha}}{A} = \frac{2-2\alpha}{2-\alpha}$$

and

$$\beta_2^{(2)} = \frac{E}{C} = \frac{A(1-\alpha)}{2A\frac{1-\alpha}{2-\alpha}} = 1 - \frac{1}{2}\alpha.$$

Setting $n = 2$ in Eq. (2) reproduces these two equations, for $k = 1, 2$.

Suppose now that Eq. (2) holds for some $n \in M$. I need to show that this hypothesis implies that the equation also holds for the next element of M, $2n$. Note that $L_{2k}^{(2n)} = L_k^{(n)}$ and $L_{2(k+1)}^{(2n)} = L_{k+1}^{(n)}$. Doubling the superscript from n to $2n$ means that we need to double the subscripts k and $k+1$ in order to identify the same two ties.

Doubling n means that between any two ties $L_{2k}^{(2n)} = L_k^{(n)}$ and $L_{2(k+1)}^{(2n)} = L_{k+1}^{(n)}$ we have a new tie, $L_{2k+1}^{(2n)}$, that lies equidistant (in real, not apparent distance) between $L_{2k}^{(2n)}$ and $L_{2(k+1)}^{(2n)}$. Using a result contained in Eq. (9) ($C = 2A\frac{E}{A+E}$), we know that the length of $L_{2k+1}^{(2n)}$ is

$$L_{2k+1}^{(2n)} = 2L_{2k}^{(2n)}\frac{L_{2(k+1)}^{(2n)}}{L_{2k}^{(2n)} + L_{2(k+1)}^{(2n)}} = 2L_k^{(n)}\frac{L_{k+1}^{(n)}}{L_k^{(n)} + L_{k+1}^{(n)}}. \tag{10}$$

For even indices, $k = 2, 4, 6, \ldots$, the relation $L_{2k}^{(2n)} = L_k^{(n)}$ implies

$$L_k^{(2n)} = L_{k/2}^{(n)}. \tag{11}$$

For odd indices, $k = 1, 3, 5, 7, \ldots$, Eq. (10) implies

$$L_k^{(2n)} = 2L_{(k-1)/2}^{(n)}\frac{L_{(k+1)/2}^{(n)}}{L_{(k-1)/2}^{(n)} + L_{(k+1)/2}^{(n)}}. \tag{12}$$

(Recall that $L_0^{(m)} = 1$ for all $m \in M$.)

With these intermediate results, we have, for $k = 2, 4, 6, \ldots$

$$\beta_k^{(2n)} = \frac{L_k^{(2n)}}{L_{k-1}^{(2n)}} = \frac{L_{k/2}^{(n)}}{\frac{2\left[L_{(k-2)/2}^{(n)}\right]\left[L_{(k)/2}^{(n)}\right]}{L_{(k-2)/2}^{(n)} + L_{(k)/2}^{(n)}}} \tag{13}$$

and for $k = 1, 3, 5, 7, \ldots$

$$\beta_k^{(2n)} = \frac{L_k^{(2n)}}{L_{k-1}^{(2n)}} = \frac{\frac{2\left[L_{(k-1)/2}^{(n)}\right]\left[L_{(k+1)/2}^{(n)}\right]}{L_{(k-1)/2}^{(n)}+L_{(k+1)/2}^{(n)}}}{L_{(k-1)/2}^{(n)}}. \tag{14}$$

The hypothesis

$$\beta_k^{(n)} = \frac{n-(n+1-k)\,\alpha}{n-(n-k)\,\alpha} \tag{15}$$

implies

$$L_k^{(n)} = \Pi_{j=1}^{k}\left(\frac{n-(n+1-j)\,\alpha}{n-(n-j)\,\alpha}\right). \tag{16}$$

For odd values of k, using Eqs. (14) and (16), we have

$$\begin{aligned}
\beta_k^{(2n)} &= \frac{\frac{2\left[L_{(k-1)/2}^{(n)}\right]\left[L_{(k+1)/2}^{(n)}\right]}{L_{(k-1)/2}^{(n)}+L_{(k+1)/2}^{(n)}}}{L_{(k-1)/2}^{(n)}} \\
&= \frac{2\left(L_{(k-1)/2}^{(n)}\right)\left(L_{(k+1)/2}^{(n)}\right)}{\left(L_{(k-1)/2}^{(n)}+L_{(k+1)/2}^{(n)}\right)L_{(k-1)/2}^{(n)}} = \frac{2\left(L_{(k+1)/2}^{(n)}\right)}{\left(L_{(k-1)/2}^{(n)}+L_{(k+1)/2}^{(n)}\right)} \\
&= \frac{2\Pi_{j=1}^{(k+1)/2}\left(\frac{n-(n+1-j)\alpha}{n-(n-j)\alpha}\right)}{\Pi_{j=1}^{(k-1)/2}\left(\frac{n-(n+1-j)\alpha}{n-(n-j)\alpha}\right)+\Pi_{j=1}^{(k+1)/2}\left(\frac{n-(n+1-j)\alpha}{n-(n-j)\alpha}\right)} \\
&= \frac{2\left(\frac{n-\left(n+1-\frac{k+1}{2}\right)\alpha}{n-\left(n-\frac{k+1}{2}\right)\alpha}\right)}{1+\left(\frac{n-\left(n+1-\frac{k+1}{2}\right)\alpha}{n-\left(n-\frac{k+1}{2}\right)\alpha}\right)} = \frac{2n-2\alpha n-\alpha+\alpha k}{2n-2\alpha n+\alpha k}.
\end{aligned}$$

Comparing the last expression with the right hand side of Eq. (2) we see that the two are identical, apart from the fact that the index n in Eq. (2) now appears as $2n$, because we have doubled the number of ties.

For k even, we have

$$\begin{aligned}
\beta_k^{(2n)} &= \frac{L_{k/2}^{(n)}}{\frac{2\left[L_{(k-2)/2}^{(n)}\right]\left[L_{(k)/2}^{(n)}\right]}{L_{(k-2)/2}^{(n)}+L_{(k)/2}^{(n)}}} \\
&= \frac{\Pi_{j=1}^{\frac{k-2}{2}}\left(\frac{n-(n+1-j)\alpha}{n-(n-j)\alpha}\right)+\Pi_{j=1}^{\frac{k}{2}}\left(\frac{n-(n+1-j)\alpha}{n-(n-j)\alpha}\right)}{2\Pi_{j=1}^{\frac{k-2}{2}}\left(\frac{n-(n+1-j)\alpha}{n-(n-j)\alpha}\right)} \\
&= \frac{1+\left(\frac{n-\left(n+1-\frac{k}{2}\right)\alpha}{n-\left(n-\frac{k}{2}\right)\alpha}\right)}{2} = \frac{2n-2\alpha n-\alpha+\alpha k}{2n-2\alpha n+\alpha k}.
\end{aligned}$$

We obtain the same result as for the case of k even. We have thus established that if the formula in Eq. (2) is valid for $n \in M$, it is also valid for $2n \in M$, recognizing that it is necessary to replace n by $2n$ in the formula. This completes the inductive proof.

Remark 1. Given the simplicity of the formula in Eq. (2), it would be astonishing if it failed for any positive integer n, not merely for the elements of M. I have restricted attention to the set M in order to produce a simple inductive proof.

Proposition 2. Using Eqs. (1) and (2)

$$\begin{aligned}
L_j^{(n)} &= L_{j-1}^{(n)}\frac{n-(n+1-j)\,\alpha}{n-(n-j)\,\alpha} \implies \\
L_j^{(n)} - L_{j-1}^{(n)} &= L_{j-1}^{(n)}\left(\frac{n-(n+1-j)\,\alpha}{n-(n-j)\,\alpha}-1\right) \\
&= L_{j-1}^{(n)}\left(\frac{\alpha}{-n+\alpha n-\alpha j}\right).
\end{aligned} \tag{17}$$

Let the real distance from L_0 to L^{anchor} be D. If L^{anchor} is the n'th tie from L_0 then the distance between ties is $\varepsilon = \frac{D}{n}$. Dividing the second line of Eq. (17) by ε gives

$$\begin{aligned}
\frac{L_j^{(n)} - L_{j-1}^{(n)}}{\varepsilon} &= L_{j-1}^{(n)}\left(\frac{\alpha}{-n\varepsilon+\alpha n\varepsilon-\alpha j\varepsilon}\right) \\
&= L_{j-1}^{(n)}\left(\frac{\alpha}{-D+\alpha D-\alpha\delta}\right),
\end{aligned}$$

where $\delta = j\varepsilon$, the distance between the first and the j'th tie. Letting $\varepsilon \to 0$ gives

$$\frac{dL(\delta)}{ds} = L(\delta)\left(\frac{\alpha}{-D+\alpha D-\alpha\delta}\right),$$

where $L(\delta) = \lim_{\varepsilon\to 0} L_{j-1}^{(n)}$. Integrating this expression, using $L(0) = 1$, gives the first part of Eq. (5). The second part of that equation uses the definition $r = -\frac{dL}{dt}L^{-1}$.

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